7.4 (continued)

multiplication of Laplace transforms

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

but $\mathcal{L}\{f(t)\}\mathcal{L}\{f(t)\} \neq \mathcal{L}\{f(t)g(t)\}$ in general,

it turns out $\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\}$

$$= \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\right\}$$

$$= F(s)G(s)$$

can show this is true by using definition and swapping order of integration

convolution integral $\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau$

$$= f(t) * g(t)$$

more to say about convolution in 7.5, 7.6

for now, it can be used as another way to do inverse transforms

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

option 1: partial fraction

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

option 2: integral property

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\} = \int_0^t \text{inverse}\,d\tau$$

option 3: convolution

$$\mathcal{L}^{-1}\left\{\underbrace{\frac{1}{s}}_{F(s)}\underbrace{\frac{1}{s^2+1}}_{G(s)}\right\} = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$f(t) = 1$

$g(t) = \sin(t)$

$$= \int_0^t (1)\sin(t-\tau)\,d\tau = \int_0^t 1\cdot\sin(\tau)\,d\tau$$

$$= -\cos(\tau)\Big|_0^t$$

$$= -\cos(t) + 1$$

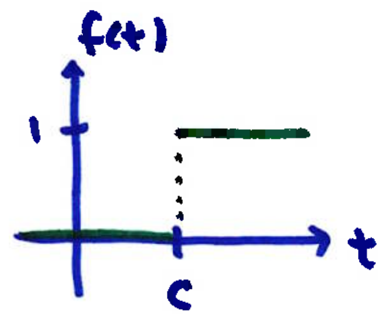
7.5 Discontinuous Input Functions

$$ay'' + by' + cy = f(t)$$

discontinuous but at least piecewise continuous

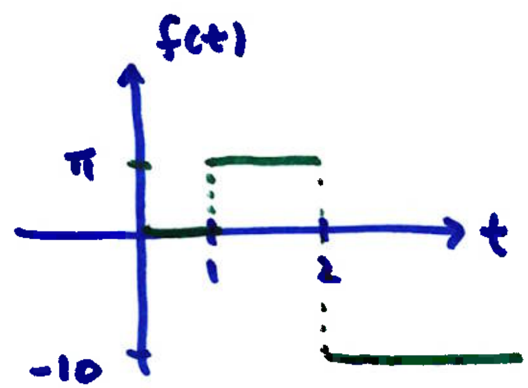
to model discontinuities, one function we can use is the unit step function

$$u_c(t) = u(t-c) = \begin{cases} 1 & \text{if } t \geq c \\ 0 & \text{else} \end{cases}$$



let's use it to model

$$f(t) = \begin{cases} 0 & 0 \leq t < 1 \\ \pi & 1 \leq t < 2 \\ -10 & 2 \leq t < \infty \end{cases}$$

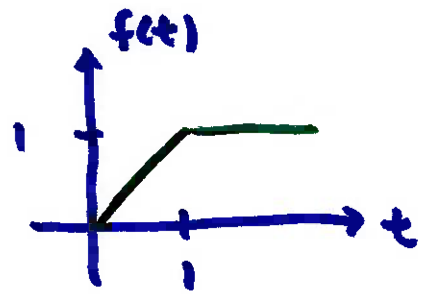


in terms of unit steps

$$f(t) = 0 + \underbrace{u_1(t)}_{\text{Switch on at } t=1}\underbrace{(\pi)}_{\text{Switch this on}} + u_2(t)\,(\overbrace{-\pi}^{\text{reset to } 0}\underbrace{-10}_{\text{want to be at this}}) = \pi u_1(t) - \pi u_2(t) - 10 u_2(t)$$

$$= \pi(u_1 - u_2) - 10 u_2$$

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$



$$= t + u_1(t)(-t+1)$$

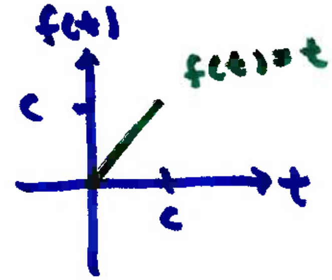
Laplace transform of unit step functions

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\, e^{-st}\,dt = \int_0^c \underbrace{u_c(t)}_{0 \text{ if } t<c} e^{-st}\,dt + \int_c^\infty \underbrace{u_c(t)}_{1 \text{ if } t \ge c} e^{-st}\,dt$$

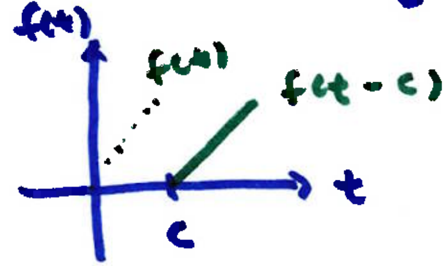
$$= \int_c^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_c^\infty = \frac{1}{s} e^{-cs}$$

$$\boxed{\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}}$$

next, $\mathcal{L}\{\underbrace{u_c(t)\, f(t-c)}_{\text{delayed activation of } f(t) \text{ by } c}\}$



delay by c



same picture shifted RIGHT by c $\rightarrow$ change t to t+c

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = \int_0^\infty u_c(t)\, f(t-c)\, e^{-st}\, dt$$

$$= \int_c^\infty f(t-c)\, e^{-st}\, dt$$

let $\tau = t-c \quad t = \tau + c$
$d\tau = dt$

$$= \int_0^\infty f(\tau)\, e^{-s(\tau+c)}\, d\tau$$

$$= e^{-sc} \int_0^\infty f(\tau)\, e^{-s\tau}\, d\tau$$

$$\boxed{\mathcal{L}\{u_c(t) f(t-c)\} = e^{-sc}\, \mathcal{L}\{f(t)\}}$$

do NOT transform f(t-c) but transform f(t)

$f(x)$

$\mathcal{L}\{u_1(t)(1-t)\}$ shift ~~RIGHT~~ LEFT by 1 → t changes to $t+1$

$= e^{-s}\mathcal{L}\{1-(t+1)\}$

$= e^{-s}\mathcal{L}\{-t\} = e^{-s}\cdot\frac{-1}{s^2}$